Calibrated Geometry and Gauge Theory on Eight-Dimensional Hyperkähler Manifolds

Maksymilian Mańko
supervised by prof. J. D. Lotay

August 2021

Abstract

The relations between calibrated geometry and Yang-Mills instantons are presented and discussed, with particular regard to 8-dimensional Hyperkähler manifolds. Calabi metric for $T^*\mathbb{CP}^n$ as well as Taub-NUT metric are discussed in 4 and 8 dimensions. The role of 2-sphere of Kähler forms admited by Hyperkähler manifold is examined and non-existence of mixed-Kähler calibrated complex submanifolds as well as of a 2-sphere of Hermitian-Yang-Mills instantons is proved. Relation to the Spin(7) geometry is examined in both contexts. Donaldson-Uhlenbeck-Yau theorem is introduced and discussed with regard to the two present instanton types. Finally, non-existence of Lagrangian instantons is explained for conditions of the Donaldson-Uhlenbeck-Yau theorem.

Contents

1 Introduction and Motivation

Calibrated geometry and gauge theory entered the scene of modern geometry in similar time. While the former stems from search of minimal submanifolds, and the latter from the study of quantum fields and particle physics, they turn out to be intimately linked. Hyperkähler manifolds were chosen to demonstrate these relations for their rich properties related to various geometries, including Kähler and Calabi-Yau. On the other hand, the dimension 8 not only provides a link to $Spin(7)$ exceptional holonomy geometry, but also allows to check whether some properties of the well-studied 4-dimensional manifolds can be extended to higher dimensions, and if there emerge new ones.

2 Background

As explained above, the Hyperkähler manifolds were chosen as objects of investigation due to multiple structures they can be endowed with. The hierarchy of smooth manifolds possessing these structures is shown in fig.1 (adapted from [1]) They will be now introduced.

2.1 Riemannian manifolds

Any smooth manifold can be endowed with a Riemannian metric [2].

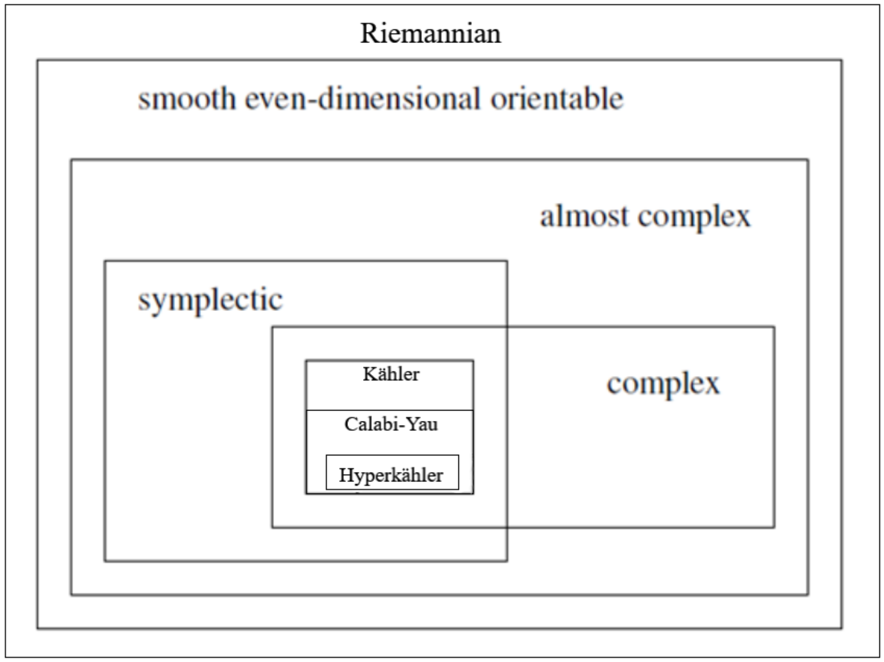

Figure 1: Hierarchy of structures on smooth manifolds

Definition 1. *Let M be a smooth manifold. The Riemannian metric g is a section of $T^*M \otimes T^*M$ (rank 2 tensor), where T^*M is the cotangent bundle. It need be positive definite, so that it assigns at each point $p \in M$ inner product on tangent vectors of M $g_p : T_pM \times T_pM \to \mathbb{R}$. Riemannian manifold is the pair (M, g).*

Any Riemannian manifold is modelled on the real space $\mathbb{R}^n$ with Euclidian metric

$$g_0 = dx_1^2 + ... + dx_n^2 \tag{1}$$

for the cotangent basis of 1-forms $dx_1, ..., dx_n$. The metric also gives the tangent vectors v the notion of norm: $|v| = \sqrt{g(v, v)}$, and so distance, and will turn out crucial in the construction of Kähler manifolds. Most of the subsequent is based on th one in [1].

2.2 Almost complex manifolds

As opposed to Riemannian, all subsequent structures are restricted only to certain even-dimensional smooth manifolds, and so is the almost complex (and complex) structure. It is, however, more convenient to be begin with its definition for vector spaces.

Definition 2. *Let V be a $2n$-dimensional vector space. A complex structure is a map $J : V \to V$, such that $J \circ J = J^2 = -\mathrm{Id}_V$*

It can be seen that the behaviour of J is meant to mimic the imaginary unit of complex numbers. By simple extension, the almost complex structure on a manifold is defined.

Definition 3. *An almost complex structure on a smooth $2n$-dimensional manifold M is a smooth field endowing each tangent space $T_pM, p \in M$ with a complex structure in the sense of Def. 2.*

This definition is the entry point to the stronger condition of complex structure.

2.3 Complex manifolds

As mentioned already, being a complex manifold is a strong condition. It can be seen in two ways. Firstly analogously to a (real) smooth manifold.

Definition 4. *Let M be a manifold of real dimension $2n$. M is said to be a n-dimensional complex manifold if it possesses the complex atlas:*

$$A = \{U_i, V_i, \varphi_i, i \in I\}$$

where I is some indexing set, $U_i \in M$ and $V_i \in \mathbb{C}^n$ are open sets, $\varphi_i : U_i \to V_i$ are holomorphic maps, such that transition functions $\varphi_j \circ \varphi_i^{-1}$ are holomorphic with an existing holomorphic inverse, i. e. biholomorphic.

Notice the main difference from the real manifold is that we require the chart maps not to only be smooth but holomorphic, which is indeed a stronger condition. An equivalent was comes from consideration of the almost complex structure.

Theorem 1. *Any complex manifold is equipped with canonical integrable complex structure J, i. e. if $\{U, \varphi\}$ and $\{U', \varphi'\}$ are overlapping complex charts with respective almost complex structures J, J', on the overlap $U \cap U'$ we have $J = J'$ [1].*

It is convenient to endow a complex chart with coordinates. These are again required to be holomorphic, i. e. holomorphic functions of some real parameter. In terms of a real double basis, analogous to the Cartesian form of complex numbers,

$$x_1, ..., x_n, y_1, ..., y_n$$

the holomorphic coordinates are given by

$$z_j = x_j + ix_j, \ j = 1, ..., n$$

The conjugate, or anti-holomorphic, basis is then

$$\bar{z}_j = x_j - iy_j$$

where index j runs as before. In a similar manner, the bases for tangent and cotangent vectors are chosen as

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}$$
$$\frac{\partial}{\partial \bar{z}_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$$

and

$$dz_j = dx_j + idy_j$$

$$d\bar{z}_j = dx_j - idy_j.$$

Note the given complex vectors, as opposed to real, are not unit, but can be easily normalised if necessary. With these conventions, the model complex structure becomes

$$J = i\begin{pmatrix} \delta_{jk} & 0 \\ 0 & -\delta_{\bar{j}\bar{k}} \end{pmatrix} \tag{2}$$

where δ_{jk} is the Kronecker delta with unbarred indexes retrieving holomorphic, and barred - anti-holomorphic coordinates.

2.4 Dolbeault splitting

The concept of holomorphic coordinates can be extended to differential forms.

Definition 5. *Let M be a complex manifold and T^*M it cotangent bundle. By complexification of the bundle we mean $T^*M \otimes \mathbb{C}$. Then there is a natural splitting into holomorphic and anti-holomorphic parts:*

$$(\pi^{1,0}, \pi^{1,0}) : T * M \otimes \mathbb{C} \to T^{1,0} \oplus T^{0,1}.$$

where $\pi^{1,0}, \pi^{1,0}$, denote respective projections. Analogously, if Λ^k denotes the space of k-forms,

$$\Lambda^k(T^*M \otimes \mathbb{C}) \cong \oplus_{l+m=k}(\Lambda^l T^{1,0} \wedge \Lambda^m T^{0,1}) := \Lambda^{l,m}$$

with projection maps constructed in the same manner.

The notion of exterior derivation follows accordingly.

Definition 6. *Let $\Omega^{l,m}$ denote the set of sections of $\Lambda^{l,m}$ and d the exterior derivative, then define Dolbeault operators as*

$$\partial = \pi^{l+1,m} \circ d : \Omega^{l,m} \to \Omega^{l+1,m}$$

$$\bar{\partial} = \pi^{l,m+1} \circ d : \Omega^{l,m} \to \Omega^{l,m+1}$$

This concept of splitting will prove useful many times later on.

2.5 Symplectic manifolds

Even-dimensional smooth manifolds may be also endowed with symplectic structure. As with almost complex structure, the principle can be introduced on a vector space.

Definition 7. *Let V be a $2n$-dimensional space. The symplectic structure is a map $\omega : V \times V \to \mathbb{R}^{(}\nvDash\ltimes)$ that is skew-symmetric bilinear and non-degenerate, i. e. if $x, y \in V$ and $\omega(x, y) = 0$, then $x = 0$ or $y = 0$.*

The model symplectic space is, naturally, $\mathbb{R}$ with symplectic double real basis $x_1, ..., x_n, y_1, ..., y_n$ and the map

$$\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{3}$$

where I_n is the $n \times n$ identity matrix. The definition for manifolds follows.

Definition 8. *Symplectic manifold is a pair (M, ω), where M is a smooth, $2n$-dimensional manifold, and ω a closed 2-form, that induces a symplectic structure on all tangent spaces $T_pM,\ p \in M$.*

The model symplectic form, on $\mathbb{R}^{2n}$ with symplectic basis is

$$\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n \tag{4}$$

The relation between the three structures described above is the topic of the next section about Kähler geometry.

2.6 Kähler manifolds

The Riemannian, complex and symplectic structures of a smooth manifold admitting them can be made interlinked to give rise to a Kähler manifold.

Definition 9. *Suppose M is a $2n$-dimensional smooth manifold, admitting a Riemannian metric g, complex form J and symplectic form ω (in this context called the Kähler form). The three are said to be compatible if for any tangent vectors u, v*

$$g(Ju, v) = \omega(u, v). \tag{5}$$

the quadruple (M, g, J, ω) comprises the Kähler manifold.

Remark 1. *Note that by positive-definiteness of the metric the Kähler form is real valued, i. e.*

$$\bar{\omega} = \omega$$

The model Kähler manifold is $\mathbb{C}^n$ with the triple of structures based on equations 1, 2, 4, adapted to real and holomorphic coordinates as before

$$g_0 = dz_1^2 + ... + dz_n^2 \tag{6}$$

$$\begin{aligned}\omega &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + ... + dz_n \wedge d\bar{z}_n) \\ &= dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n\end{aligned} \tag{7}$$

$$J = i \begin{pmatrix} \delta_{jk} & 0 \\ 0 & -\delta_{\bar{j}\bar{k}} \end{pmatrix} \tag{8}$$

The Kähler form carries and additional piece of information, namely it gives rise to a natural volume form

$$\mathrm{vol}_{\mathrm{K}} = \frac{\omega^n}{n!}, \tag{9}$$

where the exponent indicates the exterior power. This will be important in constructing Calabi-Yau manifolds, as well as in the context of calibrations.

One way of constructing a non-trivial Kähler manifold is via a scalar potential function of certain type.

Definition 10. *A smooth scalar function ρ is called strictly plurisubharmonic if for each local complex chart $\{U, z_1, ..., z_n\}$ the Hessian matrix $(\frac{\partial^2 \rho(p)}{\partial z_j \partial \bar{z}_j})$, indexes running as usual, is positive-definite at all points $p \in U$.*

Then using the Dolbeault theory one can obtain a Kähler form:

Theorem 2. *Let M be a complex manifold and ρ a plurisubharmonic function. Then*

$$\omega = \frac{i}{2}\partial\bar{\partial}\rho = \frac{\partial^2 \rho(p)}{\partial z_j \partial \bar{z}_j} dz_j \wedge d\bar{z}_k$$

is a global Kähler form. In this setting ρ is called the Kähler potential.

For full proof see [1]. This approach will turn out crucial for constructing examples of Kähler and Hyperkähler manifolds.

2.7 Calabi-Yau manifolds

There is a subclass of Kähler manifolds that deserves particular interest. It is the one of Calabi-Yau manifolds. They carry the *holomorphic volume form*

$$\Upsilon = dz_1 \wedge ... \wedge dz_n. \tag{10}$$

which also gives rise to a volume form [3]

$$\mathrm{vol}_{CY} = (-1)^{\frac{n(n-1)}{2}}(\frac{i}{2})^n \Upsilon \wedge \bar{\Upsilon} \tag{11}$$

While the above can be found on any complex manifold, Calabi-Yau have the property that eqs. 9 and 11 coincide:

$$\mathrm{vol}_K = \mathrm{vol}_{CY}. \tag{12}$$

This propery will show up again in the context of Hyperkähler manifolds. The Υ form, on the other hand, happens to be important in calibrated geometry.

2.8 Hyperkähler manifolds

We conclude this section with the class of manifolds central to this paper - the Hyperkähler manifolds. These can be thought of as "quaternionic extension" of Kähler manifolds, but being, in fact, Kähler themselves.

Definition 11. *Let M be a $2n$-dimensional complex manifold with metric g, that admits three compatible in the sense of 5 symplectic forms $\omega_I, \omega_J, \omega_K$ and complex forms I, J, K. If the latter three fulfill Hamilton relations:*

$$\begin{aligned} I \circ I &= I^2 \\ &= J^2 \\ &= K^2 \\ &= IJK \\ &= -\mathrm{Id} \end{aligned} \tag{13}$$

then the tuple $(M, g, I, J, K, \omega_I, \omega_J, \omega_K)$ is called a Hyperkähler manifold.

Remark 2. *Note that by def. 11 if a, b, c are real such that $a^2 + b^2 + c^2 = 1$, then*

$$\omega' = a\omega_I + b\omega_J + c\omega_K$$

is another Kähler form, so a Hyperkähler manifold is equipped with a 2-sphere of such forms.

It is vital to understand that every arising Kähler form corresponds to a *different* complex structure. This redefines all related concepts including imaginary unit, holomorphic functions, Dolbeault theory and holomorphic volume form. Nevertheless, it is possible to relate them to one another. One important way to do this and, in fact, an equivalent way to define a Hyperkähler manifold, is to use the fact that it admits *holomorphic symplectic form* Ω. If we distinguish the complex form I with imaginary unit i, then:

$$\begin{aligned} \Omega &= dz_1 \wedge dz_{\frac{n}{2}+1} + ... + dz_{\frac{n}{2}} \wedge dz_n \\ &= \omega_J + i\omega_K \end{aligned} \tag{14}$$

Note that for this purpose the complex dimension n was assumed to be even, rendering the Hyperkähler manifold a $4m$-dimensional manifold, for some $n =$

$2m$. This is indeed one of its important features, and stems from the fact that the triple of complex structures renders all tangent spaces isomorphic to quaternionic space $\mathbb{H}^n$ [4]. We finish this section with the relation of Ω and Υ, which is easily seen in holomorphic coordinates:

$$\begin{aligned}\Upsilon &= \frac{1}{2}\Omega^2 \\ &= \omega_J^2 - \omega_K^2 + 2i\omega_K \wedge \omega_J\end{aligned} \tag{15}$$

All the mentioned classes of manifolds can be also seen from the perspective of group theory, using holonomy, the topic of next section.

3 Holonomy

3.1 Riemannian contex and Holonomy principle

Holonomy is a manifold invariant, depending on the choice of connection and parallel transport on some vector bundle E over M. We denote it ∇ (with covariant derivative ∇_u along tangent vector u) and τ respectively, for definitions see [2]. We also say that a smooth path $\gamma : [0,1] \to M$ is a *loop based at* $p \in M$ if $\gamma(0) = \gamma(1) = p$. With this convetions, the definition is as follows

Definition 12. *The holonomy group of a vector bundle E with respect to the connection ∇ and associated parallel transport τ is*

$$Hol_p = \{\tau_\gamma \in GL(E_p) : \gamma \text{ is a loop based at p}\}$$

where $\tau_\gamma : E_p \to E_p$ is the evauluation of parallel transport around the loop map at point $p \in M$ and $GL(E_p)$ denotes the group of all symmetries of the bundle E at p.

Although the definition refers to a point on the manifold, the group is actually a global notion, henceforth denoted H and can be used to classify manifolds as we do now.

3.2 Berger's classification

Theorem 3. *(Berger, adapted from [5]) Let (M, g) be a simply connected smooth Riemannian n-dimensional manifold that is locally non-reducible (not a product space) and not locally symmetric (its group of symmetries does not contain the inversion symmetry). Then the Riemannian holonomy $H \subset SO(n)$ can be only one of possibilities below*

Remark 3. *Note that parallel tensor η means one that is preserved by the covariant derivative along any tangent vector u i. e. $\nabla_u \eta = 0$. This notion is obviously non-local, showing why holonomy is ultimately a global invariant.*

$n = \dim M$	H	Parallel tensors	Name	Curvature
n	$\mathrm{SO}(n)$	g, μ	*orientable*	
$2m$ $(m \geq 2)$	$\mathrm{U}(m)$	g, ω	*Kähler*	
$2m$ $(m \geq 2)$	$\mathrm{SU}(m)$	g, ω, Ω	*Calabi–Yau*	*Ricci-flat*
$4m$ $(m \geq 2)$	$\mathrm{Sp}(m)$	$g, \omega_1, \omega_2, \omega_3, J_1, J_2, J_3$	*hyper-Kähler*	*Ricci-flat*
$4m$ $(m \geq 2)$	$(\mathrm{Sp}(m) \times \mathrm{Sp}(1))/\mathbb{Z}_2$	g, Υ	*quaternionic-Kähler*	*Einstein*
7	G_2	g, φ, ψ	G_2	*Ricci-flat*
8	$\mathrm{Spin}(7)$	g, Φ	$\mathrm{Spin}(7)$	*Ricci-flat*

Figure 2: Berger classification of holonomy, note that in our context Ω denotes Υ for Calabi Yau manifold.

It can be seen that Berger's theorem reproduces the list of manifolds described in section 2. In particular, it is related to the multiple-of-4-dimensionality of the Hyperkähler manifolds. The *compact symplectic group* $Sp(n)$ is defined as

$$Sp(n) = Sp(2n, \mathbb{C}) \cap SU(2n)$$

where the *symplectic group over the complex field* is:

$$Sp(2n, \mathbb{C})) = \{M \in M_{2n \times 2n}(\mathbb{C}) : M^T \Omega M = \Omega\},$$

where Ω is the matrix from eq. 3.
Indeed, as this intersection $Sp(n)$ need necessarily be of multiple-of-4-dimension, and so is able to fix the holomorphic symplectic form (by fixing all three Kähler forms), rendering the manifold of the same dimension.

We round off this section with another important fact resulting from Berger's theorem - the *holonomy principle* [6].

Theorem 4. *There is a one-to-one correspondence between tensors parallel with respect to the connection ∇ and tensors fixed by the resulting holonomy group H.*

This powerful link between geometry and algebra turns out to govern some properties of calibrations, discussed later on.

4 Some Constructions of Hyperkähler manifolds

In this section we restrict attention to Hyperkähler structure in 4 and 8 dimensions. Two constructions are discussed - first one due to Calabi [7] producing a Hyperkähler metric on the cotangent bundle of complex projective space in any allowed dimension. The other, the Taub-NUT metric, occurring in 4 dimensions, is shown as an example how certain properties are lost in transition to 8 dimensions, thus making simple extensions impossible.

4.1 Calabi construction

The main idea of the method is to link the base manifold M, taken to be the projective space of complex dimension n, $\mathbb{CP}^n$ endowed with the Fubini-Study metric, to its cotangent bundle T^*M, so that a Kähler potential giving rise to a Hyperkähler manifold could be imposed.
Let M be equipped with holomorphic coordinates $z_1, ..., z_n$, with $|z|^2 = z_1^2 + ... + z_n^2$. Then the Fubini-Study metric takes the form:

$$h_{j\bar{k}} = \frac{(1+|z|^2)\delta_{j\bar{k}} - \bar{z}_j z_k}{(1+|z|^2)^2} \tag{16}$$

and can be generated from a Kähler potential

$$\Phi = \log{(1 + z^j \bar{z}_j)}, \tag{17}$$

where summation convention has been employed.
Now consider the cotangent bundle T^*M, and choose coordinates $\zeta_1, ..., \zeta_n$, such that $\{z_1, ..., z_n, \zeta_1, ..., \zeta_n\}$ is a holomorphic system of coordinates on the bundle. Additionally, define:

$$t = h(z_j, \bar{z}_k)\zeta_j\bar{\zeta}_k, \tag{18}$$

a quantity providing a link between the two coordinate sets.
Next step is to find a the holomorphic symplectic form (cf. eq. 14), and use to produce the Hyperkähler potential. In this setting, the form is

$$\Omega = dz_j \wedge d\zeta^j \tag{19}$$

and the potential

$$\Psi = \Phi \circ \pi + u \circ t \tag{20}$$

where composition with the natural projection $\pi : T^*M \to M$ simply restricts arguments of Φ to the base, and u is a solution to a scalar PDE we seek. To facilitate this task, we introduce a different non-holomorphic set of bundle coordinates $\nabla\zeta_i$ in place of ζ_j.

$$\nabla\zeta_i = d\zeta_i - \Gamma^k_{ij}\zeta_k dz_j \tag{21}$$

$$\nabla\bar{\zeta}_l = d\bar{\zeta}_l - \overline{\Gamma^k_{lm}\zeta_k dz_m} \tag{22}$$

where Γ^k_{ij} are Christoffel symbols related to the sought metric g as

$$\Gamma^k_{ij}(z, \bar{z}) = g^{k\bar{l}}\frac{\partial g_{j\bar{l}}(z,\bar{z})}{\partial z^j} \tag{23}$$

and to the Riemannian curvature tensor R as

$$R^{k\bar{l}}_{j\bar{m}} = -g^{i\bar{l}}\frac{\partial}{\partial\bar{z}^m}\Gamma^k_{ij} \tag{24}$$

The choice of coordinates in eq. 21 is dictated by the fact that since $dz_a \wedge dz_a = 0$, the Ω in eq. 19 is now

$$\begin{aligned} \Omega &= dz_j \wedge d\zeta^j \\ &= dz_j \wedge \nabla\zeta^j \end{aligned} \tag{25}$$

Using this setup and the ansatz in eq. 20, we have

$$\partial\bar{\partial}\Psi = G_{j\bar{k}}(z,\zeta;\bar{z},\bar{\zeta})dz^j d\bar{z}^k + P^{l\bar{m}}(z,\zeta;\bar{z},\bar{\zeta})\nabla\zeta_l\nabla\bar{\zeta}_l \tag{26}$$

where

$$G_{j\bar{k}} = g_{j\bar{k}}(z,\bar{z}) + (u' \circ t)R^{l\bar{m}}_{j\bar{k}}\zeta_l\bar{\zeta}_m$$

and

$$P^{l\bar{m}} = (u' \circ t)g^{l\bar{m}} + (u'' \circ t)g^{a\bar{m}}g^{l\bar{b}}\zeta_a\bar{\zeta}_b$$

The condition that construction works stands then:

$$G_{i\bar{k}}P^{j\bar{k}} = \delta_i^j \tag{27}$$

equivalent to the coupled ODEs in some auxiliary variable x

$$\begin{aligned} u'(x) + x(u'(x))^2 = 1 \text{ and } u''(x) + (1 + xu'(x)) + u'(x)(u'(x) + xu''(x)) \\ = 0 \end{aligned} \tag{28}$$

these are solved by the function:

$$u(x) = \sqrt{1+4x} - \log\left(1 + \sqrt{1+4x}\right). \tag{29}$$

For detailed treatment see last chapter of [7] Note that the fact that eq. 26 reduces to much simpler eq. 28, is a key to the dimensional universality of the construction. The discussion of the Tab-NUT metric in the subsequent section will show that it is not always the case, and then simple dimensional extensions are not possible.
To get better grasp on the construction, it is worth looking at 4- and 8-dimensional cases. Starting with the first one, the Fubini-Study metric in complex dimension 1 over coordinates $dz, d\bar{z}$ is simply:

$$h_4\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \frac{1}{(z\bar{z}+1)^2} \tag{30}$$

generated by the potential

$$\Phi_4 = \log\left(1 + z\bar{z}\right). \tag{31}$$

Now, if the bundle coordinates $\zeta, \bar{\zeta}$ are introduced, the Hyperkähler potential takes form

$$\log\left(z\,\bar{z}+1\right) - \log\left(\sqrt{\frac{4\,\xi\,\bar{\xi}}{\left(z\,\bar{z}+1\right)^2}+1}+1\right) + \sqrt{\frac{4\,\xi\,\bar{\xi}}{\left(z\,\bar{z}+1\right)^2}+1} \tag{32}$$

Note that the potential of eq. 32 is dependent only on the moduli of the variables involved, making it, and the resulting metric, radially symmetric. This is the consequence of the fact that $\mathbb{CP}$ is isomorphic to a 2-sphere.

Similar procedure can be performed for dimension 8, with coordinates $z_1, \bar{z}_1, z_2, \bar{z}_2$, $\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2$. The Fubini-Study metric is now a matrix

$$h_8 = \begin{pmatrix} \frac{z_2\,\bar{z}_2+1}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} & -\frac{z_2\,\bar{z}_1}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} \\ -\frac{z_1\,\bar{z}_2}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} & \frac{z_1\,\bar{z}_1+1}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} \end{pmatrix} \tag{33}$$

generated from the potential

$$\Phi_8 = \log 1 + z_1\bar{z}_1 + z_2\bar{z}_2 \tag{34}$$

The resulting Hyperkähler potential is now more complicated:

$$\begin{aligned}\Psi_8 &= \log\left(z_1\,\bar{z}_2 + z_2\,\bar{z}_2 + 1\right) \qquad (35)\\ &-\log\left(\sqrt{\frac{4\,\xi_1\,\bar{\xi}_1\,(z_2\,\bar{z}_2+1)}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} + \frac{4\,\xi_2\,\bar{\xi}_2\,(z_1\,\bar{z}_1+1)}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} - \frac{4\,\xi_1\,z_2\,\bar{\xi}_2\,\bar{z}_1}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} - \frac{4\,\xi_2\,z_1\,\bar{\xi}_1\,\bar{z}_2}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} + 1}\right.\\ &\left.\qquad +1\right)\\ &+\sqrt{\frac{4\,\xi_1\,\bar{\xi}_1\,(z_2\,\bar{z}_2+1)}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} + \frac{4\,\xi_2\,\bar{\xi}_2\,(z_1\,\bar{z}_1+1)}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} - \frac{4\,\xi_1\,z_2\,\bar{\xi}_2\,\bar{z}_1}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} - \frac{4\,\xi_2\,z_1\,\bar{\xi}_1\,\bar{z}_2}{(z_1\,\bar{z}_1+z_2\,\bar{z}_2+1)^2} + 1}.\end{aligned}$$

Notice arising cross-terms, signifying that the metric lost its radial symmetry. Upon increasing dimensions, one can expect the metric become more convoluted. We have seen that even in well-behaved cases, increasing dimensions from 4 to 8 leads to loss of symmetry. Now we turn to an example where it is likely impossible

4.2 Taub-NUT metric and the LeBrun's exercise

The Taub-NUT metric was developed by Taub and others, in 1950s and 1960s [8]. It is not only a Hyperkähler metric but also one of the first examples of *gravitational instanton*, an instanton (cf. section 6) which also fulfills Einstein field eqquations of General Relativity. A useful way to present it was introduced by Hawking and Gibbons, some 10 years later. Using typical convention of GR, with τ as time coordinate, $\mathbf{dx}$ the spatial part cotangent vector and $W = 1 + 2M/R$, M being the mass of the intstanton in a suitable sense and R a radial parameter, the line element is

$$ds^2 = W^{-1}(d\tau + \omega \cdot \mathbf{dx})^2 + W\mathbf{dx} \cdot \mathbf{dx} \tag{36}$$

where ω is a cotangent form such that $\mathrm{curl}\omega = \mathrm{grad}W$. This can be obtained via a Kähler potential, the *Hawking-Gibbons ansatz* given as in [9], takes the form:

$$\rho_{HG} = V(dy_1^2 + dy_2^2 + dy_3^2) + V^{-1}\eta^2, \tag{37}$$

where (y_1, y_2, y_3) is a circle fibration of $\mathbb{R}^4/\{0\}$ over $\mathbb{R}^3/\{0\}$, $V = \frac{1+4MR}{2R}$ is a function similar to W, with $R^2 = y_1^2+y_2^2+y_3^2$. Finally η is the connection 1-form (cf. section 6) for this fibration such that if $\star$ is the Hodge star $d\eta = \star_{\mathbb{R}^3} dV$. We immediately see that due to the fact that the real circle fibration of this sort is unique for the dimension 4, we need to seek a different way to extend the potential to dimension 8. For this purpose we try the *LeBrun* ansatz, introduced by C. LeBrun in [10] as an exercise.

Proposition 1. *(LeBrun's exercise) Consider a complex manifold of dimension 4, with the usual holomorphic coordinates $z_1, z_2, \ldots$. Choose non-holomorphic coordinates u, v and a real parameter m so that*

$$|z_1| = e^{m(u^2-v^2)}u$$

and

$$|z_2| = e^{m(v^2-u^2)}v$$

then

$$\rho_{LB} = \frac{1}{4}(u^2 + v^2 + m(u^4 + v^4))$$

is a Kähler potential giving rise to a metric isomorphic to the Taub-NUT metric.

here we sketch the proof given by Auvray in [9]

Proof. The idea is to exploit the property of the Kähler form described n previous sections:

$$\omega^2 = 2\mathrm{vol}_4 = \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

This is to say that if the ω obtained from $\rho_L B$ squares to the volume form, it need be Kähler. This is checked to be true, and the key point is that coordinate assignment of prop. 1 leads to $uv = |z_1 z_2|$, which allows to eliminate non-holomorphic coordinates in favour of holomorphic ones. Next step is to retrieve the triple (y_1, y_2, y_3) from u, v and holomorphic coordinates. This is done by

$$y_1 = \frac{1}{2}(u^2 - v^2),\ y_2 = \Im(z_1 z_2),\ y_3 = -\Re(z_1 z_2).$$

Finally setting V and R as it was earlier and writting η as

$$\eta = \frac{i}{4R}(u^2(\frac{d\bar{z_1}}{\bar{z_1}}) - v^2(\frac{d\bar{z_2}}{\bar{z_2}})),$$

eq. 37 is retireved. □

We now try to extend the LeBrun's ansatz to 8 dimensions, noting that to mimic the proof we now need to raise the obtained ω to 4th power. The first natural ansatz to try, for holomorphic z_j, j up to 4, and non-holomorphic u_j is

$$|z_j| = e^{u_j^2 - u_i u^i} u_j, \tag{38}$$

but we note immediately that necessary $u_1...u_4 = |z_1...z_4|$ is not satisfied, and indeed direct computation show that the retrieved ω is not even Kähler. A solution to this is the following cyclic ansatz, so that exponents telescope upon multiplication

$$|z_1|= e^{u_1^2-u_2^2}u_1, ..., |z_4|= e^{u_4^2-u_1^2}u_4. \tag{39}$$

This, however, fails as well. We see that although the initial identity is fulfilled, raising ω to the 4th power in more variables generates other types of cross terms, say $u_1^3 u_2$. Is is easily seen that there are far more cross-terms present, and hence identities needed, then there are assigment equations. Direct computation, on the other hand, shows that there is no sufficient cancelations to make the problem simpler. Therefore we can conclude that this ansatz is not realisable in dimension 8, and likely in any higher. This is an interesting example of how different the behaviour of equations in dimension 4 is, in this case due to a fact as simple as that there is only one cross-term in the binomial expansion of order 2. While there are certainly ways to extend Taub-NUT metric beyond dimension 4 (by perhaps replacing the circle fibration from eq. 37 with some generalisation of Hopf fibration), it is unlikely that it indeed is *the same metric* in the sense as in the Calabi's $\mathbb{CP}^n$ construction.

5 Calibrated geometry

We now leave intricacies of 8-dimensional Hyperkähler geometry, to introduce a seemingly unrelated field - calibrated geometry.

5.1 Motivation and basic definitions

The idea was conceived in the 1980s by Harvey and Lawson [11], to probe minimal surfaces, or, more broadly, manifolds. Minimal submanifold is one that locally extremises the volume functional, slightly counter-intuitively to the name.

Definition 13. *Let M be a manifold and N its submanifold. N is said to be minimal if*

$$\frac{d}{dt}\mathrm{vol}(F(S,t))|_{t=0}= 0$$

for all variations F with compact support $\bar{S}$ dependent on F.

Note this problem is described by 2nd order PDEs. Calibrations, somewhat similar to the variational methods applied to Newtonian physics, reduces this to a 1st order PDE. To achieve this, we introduce the following definition.

Definition 14. *(Harvey-Lawson) A differential k-form form η on a Riemannian manifold (M, g) is a calibration if*

1. *$d\eta = 0$ and*

2. $\eta(e_1, ..., e_n) \leq 1$ *for all unit tangent vectors* $e_1, ..., e_k$ *on* M

It is natural to ask when $\eta = 1$.

Definition 15. *If* η *is a calibration* k*-form on the Riemannian manifold* (M, g)*, and oriented* k*-dimensional submanifold* N *is calibrated by* η *if* $\eta|_N = \mathrm{vol}_N$*, or, equivalently,* $\eta(e_1, ..., e_k) = 1$ *for an oriented orthonormal basis vectors* $e_1, ..., e_k$ *on* T_pM *for all* $p \in M$

We are now in position to state the main theorem for calibrated submanifolds.

Theorem 5. *Let* N *be a calibrated submanifold. Then* N *is minimal, and moreover if* F *is any variation with compact support* $\bar{S}$ *then* $\mathrm{vol}(F(S,t)) \geq \mathrm{vol}(S)$*, i. e.* N *is volume minimising.*

for the proof see [12]. Note that this is a stronger result than minimality, since it also provides volume minimisation.
While calibrations turn out to be relatively abundant, for instance any bounded form on $\mathbb{R}^n$ can be rescaled to become a calibration, non-trivial (not planes) calibrated manifolds are less obvious. In particular, not every calibration possesses a corresponding submanifold. While there is no strict way to determine it, the holonomy principle of theorem 4 gives one hint. It is often the case, as shown in subsequent sections that if a calibartion is one of the parallel tensors fixed by the holonomy group it is likely to have a calibrated submanifold, and Hyperkähler manifolds provide several of those.

5.2 Kähler calibrations

First example of those comes from the Kähler form, fixed by the holonomy group $U(n)$ The result is as follows

Theorem 6. *On a Kähler manifold* (M, g, J, ω)*, the form* $\frac{\omega^k}{k!}$ *is a calibration for any integer* $k \leq n$*. Moreover, each form has a* k*-dimensional* J*-complex calibrated submanifold* N*, i. e. for all points* $p \in N$ $J(T_pN) = T_pN$*.*

The argument, following closely one in [12], is nevertheless worth mentioning, as certain techniques used are common for calibrations discussed further on. The first step is the *Wirtinger's inequality.*

Theorem 7. *Let* ω *be the Kähler form on a* n*-dimensional complex manifold, and a set of unit tangent vectors on* T_pM *at point* $p \in M$*,* $e_1, ..., e_k$*,then:*

$$\frac{\omega^k}{k!} \leq 1.$$

We need the following simple lemma for the proof.

Lemma 1. *Let* η *be a calibration. If* $\star\eta$ *is closed, then it is a calibration.*

Proof. We know $\star\eta$ is closed, so we only need to prove it is smaller or equal to 1. Recall that by definition of the Hodge dual $\det(\eta \wedge \star\eta) =< \eta, \eta >\leq 1$ [13], where the inner product refers to the Gram determinant:

$$< \alpha, \beta >= \det(< \alpha_i, \beta_j >^k)$$

for k-forms α, β. This is invariant upon action of the Hodge dual, so $|\eta|= |\star\eta|$. The result follows. □

We now proceed to prove the main theorem.

Proof. Firstly, we see that $|\frac{\omega^k}{k!}|^2= \frac{n!}{k!(n-k)!}$, and by relation 9, so $\star\frac{\omega^k}{k!} = \frac{w^{n-k}}{(n-k)!}$, and by the lemma, we can restrict attention to the case $k \leq \frac{n}{2}$.
Next, an important device to use is the canonical form of the complex plane $\mathrm{Span}\{e_1 \wedge Je_1 \wedge ... \wedge e_n \wedge Je_n\}$. To see why any plane P can be expressed this way, in terms of some basis e_i, consider unit two tangent vectors $u, v \in P$. Then, $< Ju, v >$ need have a maximum, say $\cos\theta_1 =< Ju, v >$ for $0 \leq \theta_1 \leq \frac{\pi}{2}$. Now consider a unit vector $w \in P$ orthogonal to the span of u and v. The function

$$f_w(\theta) =< Ju, \cos\theta v + \sin\theta w >$$

has a maximum at $\theta = 0$ so $f'_w(0) =< Ju, w >= 0$. By the same token, $< Jv, w >= 0$ and so $w \in \mathrm{Span}\{u, v, Ju, Jv\}$.
There are now two cases. If $\theta_1 = 0$, then $v = Ju$, so choose $u = e_1$ and $v = Je_1$, rendering $P = \mathrm{Span}\{e_1, Je_1\} \times \mathrm{Span}\{e_1, Je_1\}$.
If $\theta_1 \neq 0$, set $u = e_1$, $w = e_2$, so that $P = \mathrm{Span}\{e_1, e_2, Je_1, Je_2\}\times\mathrm{Span}\{e_1, e_2, Je_1, Je_2\}^{\perp}$.
Proceeding by induction, we acquire and oriented basis $\{e_1, Je_1, ..., e_n, Je_n\}$ for $\mathbb{C}^n$, so that:

$$P = \mathrm{Span}\{e_1, \cos\theta_1 Je_1 + \sin\theta_1 e_2, ..., e_{2k-1}, \cos\theta_k Je_{2k-1} + \sin\theta_k e_{2k}\}$$

where $0 \leq \theta_1 \leq ... \leq \theta_{k-1} \leq \frac{\pi}{2}$ and $\theta_{k-1} \leq \theta_k \leq \pi - \theta_{k-1}$.
Now, $\omega = \Sigma_{j=1}^n e_j \wedge Je_j$, so $\frac{\omega^k}{k!}$ restricts to P as a product of $\cos\theta_j$, certainly less or equal to 1. Furthermore, equality holds only if all angles are zero, meaning P is complex. □

Remark 4. *Note that the choice of θ_k is restricted compared to the other angles. This is due to the fact that each time a θ_i is to be introduced, there is a choice of orientation, as there still are vacant dimensions. In the last case, however, the orientation is already determined, thus the angles are different.*

The canonical form of the complex plane will recur in description of further calibrations. This category of calibrations will be also crucial in defining the Yang-Mills instantons. We now look into the case when our underlying manifold is also Hyperkähler.

5.3 Mixed Kähler calibrations

Recall from previous sections that Hyperkähler manifold admits three distinguished Kähler forms $\omega_I, \omega_J, \omega_K$, giving rise to a 2-sphere of Kähler forms. Theorem 5.2, endows each of the forms (and their exterior powers) with corresponding calibrated submanifolds. It is natural to ask, whether all these are related. We answer this question for a case when two calibrated submanifolds, say P and Q, corresponding to different Kähler forms ω_I and ω_J have a non-empty intersection $P \cap Q$, where immediately we see there is $\frac{\omega_I^k \wedge \omega_J^l}{k!l!} = 1$, which we call the mixed-Kähler calibration. We prove the following result

Proposition 2. *In the setup described above, the intersection of two complex calibrated submanifolds is not a complex submanifold.*

Proof. Consider first real dimension 2 (complex 1), i. e. $N = P \cap Q$ is spanned by a basis of two vectors. We are going to prove that under certain circumstances, it can be shown that we construct a complex manifold with respect to a different complex structure. Basing on previous discussion, consider complex charts $U \subseteq P$ and $V \subseteq Q$, spanned by bases e, Ie and f, Jf respectively, such that N belongs to both Span$\{e \wedge Ie\}$ and Span$\{f \wedge Jf\}$. To investigate geometric relation of the two, we wish to rotate one basis into the other, using some orthonormal matrix A. Note that A need not be scalar, and may contain operators as entries. Explicitly:

$$\begin{aligned} f &= A_{11}e + A_{12}Ie \\ &Jf \\ &= A_{21}e + A_{22}Ie \end{aligned} \tag{40}$$

Now, we act with J on f and equate the two equations, eliminating f

$$A_{11}Je + A_{12}Ke = A_{21}e + A_{22}Ie \tag{41}$$

where we used $J \circ I = -K$. Equating coefficients, and rearranging gives:

$$\begin{aligned} A_{21} &= L \\ &= -A_{22}I + A_{11}J + A_{12}K. \end{aligned} \tag{42}$$

so L is, in fact, a valid complex structure with respect to e. Since all vectors are, as linear combinations of the two bases, within U and V, we find that there exists a L-complex curve spanned by e, Le, belonging to $U \oplus V$. Then since at each point $p \in N$ we have $L(T_pN) = T_pN$ by construction, and by assumption $\frac{\omega_I^k \wedge \omega_J^l}{k!l!} = 1$. However, this is by no means a calibrated submanifold, as the mixed Kähler form is at least dimension 4.

Therefore we need $\dim(P \cap Q) \geq 4$. In this case, we again choose two bases, this time running as $e^1, Ie^1, ..., e^p, Ie^p$ and $f^1, Jf^1, ..., f^q, Jf^q$. The indexes are chosen to as "high" not "low" to invoke the summation convention more easily. Then the rotation takes form:

$$\begin{aligned} f_p &= A_{pq}e^q + A_{p\bar{q}}Ie^q \\ &Jf_p \\ &= A_{\bar{p}q}e^q + A_{\bar{p}\bar{q}}Ie^q \end{aligned} \tag{43}$$

where we think of barred and unbarred indexes as in the symplectic form of eq. 3. We now see that the argument of dimension 2 leads us to a situation where each vector e_q is acted upon by a different column of the matrix A having complex forms as entries, so we are unable to extract a consistent complex form L, unless A is diagonal with all terms equal, i. e. the two bases are parallel. We conclude that beyond that there are no mixed-Kähler complex calibrated submanifolds.

□

Note that this does not exclude the possibility of existence of any mixed-Kähler calibrated submanifolds, for any dimension. However, equally, they might not exist at all, since our argument assumed non-empty intersection of the parent manifolds. One hint is that the mixed-Kähler calibration, by the holonomy principle (theorem 4) are all fixed by the $Sp(n)$ of corresponding dimension, which could favour the existence of calibrated submanifolds.
We now proceed in the hierarchy of structures to the Calabi-Yau manifolds.

5.4 Lagrangian calibrations

Recall that Calabi-Yau manifolds are equipped with the holomorphic volume form Υ (eq. 10), fixed by the holonomy group $SU(n)$. This form gives rise to a family of calibrations related to Lagrangian submanifolds of symplectic geometry [1]. The theorem, again based on [12], is as follows.

Theorem 8. *If M is a Calabi-Yau manifold with the holomorphic volume form Υ. Then $\Re(e^{-i\theta}\Upsilon)$ is a calibration for any $\theta \in \mathbb{R}$.*

the related, and sufficient to prove theorem 8 as $d\Upsilon = 0$ is the following.

Theorem 9. *On $\mathbb{C}^n$, $|\Upsilon(e_1, ..., e_n)|$ for any unit vectors $e_1, ..., e_n$, if and only if P=Span$\{e_1, ..., e_n\}$ is a Lagrangian plane, i.e. a plane such that $\omega|_P \equiv 0$.*

The proof again relies on the canonical form of the complex plane.

Proof. Let $e_1, ..., e_n$ be the standard basis for $\mathbb{R}^n$, and P a n-plane in $\mathbb{C}^n$. There exists a matrix $GL(n, \mathbb{C})$ so that $f_1 = Ae_1, ..., f_n = Ae_n$ is an orthonormal basis for P. Then $\Upsilon(Ae_1, ..., Ae_n) = \det_{\mathbb{C}}(A)$. Then we have:

$$|\Upsilon(f_1, ..., f_n)|^2 = |\det_{\mathbb{C}}(A)|^2 = |\det_{\mathbb{R}}(A)| = |f_1 \wedge Jf_1 \wedge ... \wedge f_n \wedge Jf_n| \leq |f_1||Jf_1|...|f_n||Jf_n| = 1$$

with equality only if $f_1, Jf_1, ..., f_n, Jf_n|$ are orthonormal. Now if we recall the compatibility condition of Kähler manifolds in eq. 5, this is equivalent to $\omega|_P \equiv 0$ if and only if $JP = P^\perp$ □

The submanifolds where the latter condition holds are called special Lagrangians of phase $e^{i\theta}$, and are always half the dimension of the manifold. It will be shown later that, as opposed to Kähler and Cayley calibrations, these are not known to be related to instanton connections.

5.5 Cayley calibrations

Recall the Berger's classification of manifolds by the notion of holonomy in theorem 3. The $Spin(7)$ is a 28-dimensional group coinciding with the universal cover of $SO(7)$. A $Spin(7)$ manifold, one of holonomy $Spin(7)$, is an 8-dimensional manifold, equipped with a 4-form Φ called the Cayley form fixed by the group. Moreover, it is know that $Sp(4) \subset SU(4) \subset Spin(7)$ and so an 8-dimensional Hyperkähler manifold can be equipped with an analogue of Cayley form, as shown by [14]. More explicitly, if we distinguish the pair (I, ω_I), the form in question is:

$$\Phi = -\frac{1}{2}\omega_I^2 - \frac{1}{2}\omega_J^2 + \frac{1}{2}\omega_K^2. \tag{44}$$

By a proposition in [14], for an 8-dimensional Hyperkähler manifold, this form is fixed exactly by $Spin(7)$.

Remark 5. *In dimensions other than 8, the fixing group would be $Sp(n) \cdot Sp(1)$, which is also the holonomy group of quaternionic-Kähler manifolds, as seen in fig. 3. Note well these manifolds are not Kähler.*

To prove that Φ is in fact calibration, we use the following relation between the holomorphic volume form Υ (eq. 10) and holomorphic symplectic form Ω (eq. 14).

$$\begin{aligned} \Upsilon &= \frac{1}{2}\Omega^2 \\ &= \frac{1}{2}(\omega_J^2 - \omega_K^2 + 2i\omega_J \wedge \omega_K) \end{aligned} \tag{45}$$

so that we have

$$\Phi = -\frac{1}{2}\omega_I - \Re(\Upsilon). \tag{46}$$

Now we state the main theorem of this section.

Theorem 10. *Let M be an 8-dimensional Hyperkähler manifold. Then the Cayley form Φ is a calibration. Moreover, for a unit 4-form ξ, $\Phi(\xi) = 1$ if and only if ξ belongs to some quaternionic plane over M (by canonical association of tangent spaces of M with $\mathbb{H}$), and $Span\{\xi\}$ is an oriented 4-plane in the space in octonions $\mathbb{O}$, such that $\Phi(\xi) = |\xi|$.*

The full proof of this is the subject of [14], and here we only give a sketch.

Proof. First step is that $\Re(\Upsilon)(\xi) = 0$ if and only if $\mathrm{Span}\{\xi\}$ contains a complex line spanned by $e \wedge Ie$ for some vector e. To see this assume $\xi = e \wedge Ie \wedge u \wedge v$ for some vectors u, v. This necessarily has a (1, 1) component, so is at most a

(3, 1) form, while $\Re(\Upsilon)(\xi)$ is a (4, 0) form, the result follows.
Then the idea is similar to the one in proof of 5.2, to cast

$$\xi = e_1 \wedge (\cos\theta_1 I e_1 + \sin\theta_1 e_2) \wedge e_3 \wedge (\cos\theta_2 I e_3 + \sin\theta_2 e_4)$$

with the same angle restrictions as there. Using the result of first step, we see that in Φ the only surviving terms are

$$\Phi(\xi) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \Re(\Upsilon)(e_1 \wedge e_2 \wedge e_3 \wedge e_4).$$

We now $\Re(\Upsilon)$ is calibration, thus $\Phi(\xi) \leq 1$ as desired.
Moreover, $\Phi(\xi) = 1$ forces $\Re(\Upsilon) = 1$, so $\mathrm{Span}(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \subset V$ for some quaternionic plane $V \subset \mathbb{H}^n$. Since the real span of ξ is contained in $\mathbb{H}$-span, it follows that indeed ξ lies in a quaternionic plane. The final part regarding relation to octonions is much more involved and can be found in the mentioned paper. □

As explained before, this family of calibrations is exclusive to the 8-dimensional Hyperkähler manifolds. This will be linked to existence and nature of $Spin(7)$ instantons described in the next section about gauge theory.

6 Gauge theory

Gauge theory is central to modern physics, especially as the theoretic basis of the Standard Model of particles. It has been also used as an effective tool to study connections on principal bundles [15]. We start by defining the environment where the Yang-Mills equations could be defined.

6.1 Principal G-bundles and connections.

A principal bundle over a manifold M is an object similar to the vector bundle, and while the latter is meant to locally resemble a product space $M \times V$ for a vector space V, the former is related to $M \times G$, where G is a group inducing an action on the manifold. Formally, the definition is as follows [15]:

Definition 16. *A principal G-bundle for some topological group G is a fibre bundle E with projection $\pi : E \to M$, together with a continuous, free and transitive group action $E \times G \to E$ preserving the fibres. In particular each fibre is homeomorphic to the group itself.*

Note that by this definition, as vector bundle associates a vector space to each point on the manifold, a G-bundle associates to each point the corresponding orbit of the group action.

Remark 6. *While G-bundles appear much distinguished from the vector bundles, a G-bundle might be thought of as a vector bundle using the notion of representations.*
Let E be the $G-bundle$ and $\phi : G \to GL(V)$ be any representation (group

homomorphism) on some vector space V. The action of $GL(V)$ on V produces as orbits all oriented bases of V, thus giving rise to the frame bundle - principal $GL(V)$-bundle. There the fibres $\pi^{-1}(p) \equiv V$. This allows to associate the fibres, being orbits of the group action, to vector spaces. In case of the frame bundle, they can be naturally associated to tangent spaces. For general G-bundle, using ϕ, we obtain a notion of a vector bundle with structure group G, equivalent in the above sense to the G-bundle. Some authors, e. g. Tian [3], use both notions interchangeably.

We will mainly focus on the $SU(4)$ in subsequent discussion.

6.2 Yang-Mills equations

Objects of our primary interests are Yang-Mills instantons, which are connection on the G-bundle. Explicitly, a connection A is a $\mathfrak{g}$-valued 1-form, or $A \in \mathfrak{g} \otimes T^*M$, where $\mathfrak{g}$ is the Lie algebra associated to G. The connection is realised by defining a covariant derivative $d_A\eta = d + \frac{1}{2}[A, \eta]$, for any form η.
Any connection has an associated curvature 2-form $F_A = d_A A = d_A + A \wedge A$, which turns out to be a tensor, as opposed to A.

Remark 7. *The meaning of F_A becomes clear upon a choice of coordinate frame X_i [16]. Then*

$$(F_A)_{ij} = [\nabla_{X_i}, \nabla_{X_j}]$$

where ∇_{X_i} is the covariant derivative in the direction of the coordinate direction vector X_i and $[,]$ is the Lie bracket. Then this also gives an approximation to corresponding parallel transport over an infinitesimal parallelogram of side δ:

$$T_{ij} = 1 + (F_A)_{ij}\delta^2 + O(\delta^4)$$

We can also define a dual of d_A by $d_A^* = - \star d_A \star$. Then we are in position to state the following.

Definition 17. *A connection A is Yang-Mills, if $d_A^* F_A = 0$. Then also, by Bianchi identity, $d_A F_A = 0$.*

Yang-Mills connections are critical point of the Yang-Mills functional, introduced in the next section. Its minima, need more restrictions to be found, which motivates the following definition.

Definition 18. *For a 4-dimensional manifold, a A connection A is an Anti Self-Dual (ASD) Yang-Mills instanton if $F_A = - \star F_A$. This also implies $d_A^* F_A = 0$, so the connection is also Yang-Mills.*

There are multiple examples of instantons in dimension 4, including the complex projective plane with Fubini-Study metric and the Taub-NUT solution, discussed in previous section. The latter, as already mentioned, additionally satisfies field equations of General Relativity, and is hence called a gravitational instanton.
There is a natural extension of definition 18 to higher dimensions, surprisingly enough involving calibrations, but before that we introduce the Yang-Mills functional.

6.3 Yang-Mills functional

Tracing back the physical origin of gauge theories, they describe physical fields interacting with particles, first one to be described like this being electromagnetic field. Such setting often leads to a notion of energy stored in the field, and in Yang-Mills theory this takes form of the Yang-Mills functional [12].

Definition 19. *For the connection A over manifold M, the Yang-Mills functional is defined as*

$$\begin{aligned} YM(A) &= \frac{1}{4\pi^2}\int_M |F_A|^2 \mathrm{vol} \\ &= ||F_A||^2 \end{aligned} \tag{47}$$

Remark 8. *The Yang-Mills equations can actually be derived from the functional, by applying the principle of least action. Then they are just the corresponding Euler-Lagrange equations.*

What is interesting, the Yang-Mills functional depends on the topology of the bundle rather than the structure of the manifold. In particular, notice that eq. 47 can be split into self-dual and anti-self-dual parts [12]

$$\begin{aligned} YM(A) &= ||F_A||^2 \\ &= ||F_A^+||^2 + ||F_A^-||^2 \textit{ and } ||F_A^+||^2 - ||F_A^-||^2 \\ &= \int_M \mathrm{tr}(F_A \wedge F_A) \\ &= \kappa(E) \end{aligned} \tag{48}$$

where $\kappa(E)$ is a Chern-Weil topological invariant. Then

$$\begin{aligned} YM(A) &= 2||F_A^+||^2 - \kappa(E) \\ &= 2||F_A^-||^2 + \kappa(E) \end{aligned} \tag{49}$$

where we choose $\kappa(E) \leq 0$ so that the ADS instanton produce minima.
The above discussion can be extended to higher dimensions, in a way closely related to calibrations.

6.4 Generalisation beyond dimension 4

The definition we gave for instanton so far was restricted to dimension 4. Since we are interested in dimension 8, we consider the following generalisation:

Theorem 11. *Tian For a compact Lie G-bundle E over an n-dimensional manifold M, and closed form Θ of dimension $n-4$, then if*

$$\Theta \wedge F_A = - \star F_A$$

then $d_A^* F_A = 0$, so A is a Yang-Mills connection. Moreover, if M is compact with boundary then

$$YM(A) = (2c_2(E) - \frac{r-1}{r} c_1(E)) \cdot [\Theta],$$

where c_1, c_2 are 1st and 2nd Chern classes of the bundle, r is its rank, and $[\Theta]$ is the cohomology class of Θ.

Full proof can be found in [3].
Our main consideration concentrate on the unitary bundle $SU(4)$. Since $c_1(SU(4)) = 0$, the Yang-Mills functional reduces to

$$YM(A) = 2c_2(E) \cdot [\Theta] \tag{50}$$

Armed with this we now proceed to consider different choices of Θ for 8-dimensional Hyperkähler manifolds.

6.5 Hermitian Yang-Mills instantons

First one on the list is the square of the Kähler form $\Theta = \frac{\omega^2}{2}$:

$$\frac{1}{2}\omega^2 \wedge F_A = - \star F_A \tag{51}$$

Since we work on a unitary bundle, the curvature tensor has a natural splitting:

$$F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2} \tag{52}$$

where upper indexes indicate corresponding parts of the form. The equivalent set of equations turns out to be:

$$F_A^{0,2} = 0 \text{ and } F_A^{1,1} \cdot = \lambda \mathrm{Id} \tag{53}$$

where

$$\lambda = \frac{nc_1(E) \cdot [\omega]^{n-1}}{r[\omega]^n}$$

which turns out to be 0 for $SU(4)$. Also note that $F_A^{0,2} = 0$ implies $F_A^{2,0} = \overline{F_A^{0,2}} = 0$. The connections satisfying this set of equations are called the *Hermitian Yang-Mills (or HYM) instantons*, and are minimisers of YM. Full discussion can be found in [3].
In previous sections we mentioned that on a Hyperkähler manifold, there is a 2-sphere of Kähler forms. A natural question to ask then is whether we are able to produce a corresponding set of HYM instantons using eq. 53. The answer turns out to be negative, as we prove now.

Proposition 3. *Let E be a unitary bundle over with the corresponding Lie algebra $\mathfrak{g}$, over a Hyperkähler manifold. Then the set of all Hermitian-Yang-Mills instantons connections is not closed under the action of $SO(3)$.*

Proof. First recall the types of objects from eq. 51. $\omega \in \Lambda^2 T^*M$ is a regular covariant 2-form over M, $F_A \in \mathfrak{g} \otimes \Lambda^2 T^*M$, is a Lie algebra-valued 2-form.
We also know that $SO(3)$ acts linearly by rotations as $\circ : SO(3) \times \Lambda^2 T^*M \to \Lambda^2 T^*M$ on the space of 2-forms, of which the subset K of all admissible Kähler forms of M is invariant under this action. We now wish to understand whether this action distributes over the wedge product. Recall that the wedge product for k-forms ξ, η is defined as

$$\xi \wedge \eta = \sum_{\sigma \in S_k} \text{sign}(\sigma) \xi_{\sigma(1)} \otimes \eta_{\sigma(2)},$$

where σ are permutations in the symmetric group S_k, and $\text{sign}(\sigma)$ is the sign of the permutation. On the other hand, by the universal property of tensor product, the $SO(3)$ action should distribute over it, as it does for the regular Cartesian product. We thus conclude that the action distributes over the wedge and for some $a \in SO(3)$ we have

$$a \circ (\frac{1}{2}\omega^2 \wedge F_A) = (a \circ \frac{1}{2}\omega^2) \wedge (a \circ F_A) = -a \circ (\star F_A).$$

Finally, recall, from properties of Hodge dual, that for a vector $v \in V$ for some vector space, then $\star v \in V^\perp$.
Now, assume towards contradiction that there is a set of instantons I in one-to-one correspondence with K. Then, by the distributivity of the group action, so that as $\omega \in K$ and $F_A \in I$, I is $SO(3)$-closed. But then, we would need $\star I$ to be $SO(3)$-closed as well. However, the orthogonal complement of a 2-sphere in the ambient, higher dimensional set is not a sphere, therefore we have a contradiction. The result follows. □

The argument does not forbid the set of admissible instantons to be closed under some group action, but this is certainly not related to the 2-sphere of Kähler forms. We have seen it from the sheer consideration of spaces involved. Another perspective will be provided in the following section.

6.6 Stable bundles and Donaldson-Uhlenbeck-Yau theorem

In this section we present a very strong theorem on the existence of Hermitian Yang-Mills instantons for certain bundles, provided firstly for Riemannian surfaces by Donaldson, then generalised to manifolds by Uhlenbeck and Yau [17] in the 1980s. To state it we need two the notions of sheaves and stable bundles. While the exact definitions of the two are much beyond the scope of this paper, we give the intuition here.
The notion of stability is, in principle, related to potential in physics, in the sense we find an object which maximises the related 'slope'. Here by the slope μ we mean

$$\mu(E) = \frac{\text{deg E}}{r}, \tag{54}$$

where deg E $= \int_M c_1(E) \wedge \star\omega$, is called the degree of the bundle, dependent on the topology of E, and r is again its rank. The objects which are to be used are subsheaves $\mathcal{F}$ of the structure sheaf $\mathcal{E}$ of the bundle. We require that for every proper subsheaf $\mathcal{F}$ of $\mathcal{E}$

$$\mu(\mathcal{F}) < \mu(E). \tag{55}$$

Note that this, due to the definition of slope, is a topological condition. This relates to the fact the Yang-Mills is dependent on the topology of the bundle, and thus minimising it to obtain the Equipped with this intuition, we state the theorem

Theorem 12. *(Donaldson-Uhlenbeck-Yau) A stable holomorphic vector bundle over a compact Kähler manifold admits a unique Hermitian-Yang-Mills connection.*

The proof is the subject of [17]. Note that $SU(4)$, having trivial c_1, is both stable and has natural holomorphic structure over a complex manifold.
It is worth considering what this means in the context of Hyperkähler manifolds, which can be thought of as carrying multiple Kähler structures at once. It would be intuitive to think that is the bundle admits a HYM connection with respect to one complex structure, say I, then it is likely to do so for J, K, as well, since the three behave similarly. While it might work for a model manifold, like $\mathbb{H}^n$ or $\mathbb{C}^{2n}$, it most likely will not for general one. The reason is that the property of being holomorphic (and possibly stable) depends on the choice of the complex structure, and all its consequences, including imaginary unit or transition maps. Thus, while existence of I-HYM connection likely excludes others, even if they do exist the theorem is not violated, as the setup changes.
As mentioned before, this provides another perspective on the question of proposition 3. While the Kähler forms may be manipulated together in various ways, the instantons, requiring more specific set of assumptions, cannot. Although the equation related one to another appears simple, abstract construction of the latter does not convey symmetries of the former. We now consider another family of instantons, which, quite suprisingly, in our setup turn out to be closely related the HYM connections.

6.7 Spin(7) instantons

As explained in previous sections, 8-dimensional Hyperkähler manifolds have strong ties to the exceptional $Spin(7)$ geometry. Of course, there is no obstruction to consider eq. 11 in the $Spin(7)$ setup, using the Cayley form fixed by $Spin(7)$, Φ in place of Θ.

$$\Phi \wedge F_A = - \star F_A \tag{56}$$

It was proved by Lewis [18], that this equation indeed has solutions. Moreover, in the same paper, another result is proved:

Theorem 13. *Let E be over a manifold M, with holonomy contained in $SU(4)$. Suppose that E admits a Hermitian-Yang-Mills connection, then any $Spin(7)$*

instanton (in the sense of eq. 56) on the bundle E is also a Hermitian-Yang-Mills connection.

Remark 9. *Note that, if the bundle E also admits the unique HYM connection in the sense of the Donaldson-Uhlenbeck-Yau theorem, the equations of $Spin(7)$- and HYM- instantons are necessarily equivalent. This forces the 8-dimensional manifolds with suitable bundles to admit $Spin(7)$-instanton.*

While the proof is rather involved and can be found in [18], the form of the Cayley calibration on a 8-dimensional Hyperkähler manifold (eq. 46) gives some intuition. The $Spin(7)$-instanton equation in this context takes form

$$(\frac{1}{2}\omega_I + \Re(\Upsilon)) \wedge F_A = - \star F_A \tag{57}$$

Now, a useful corollary can be drawn for HYM instantons. We established that the curvature tensor of a HYM instanton is a (1, 1)-form. A 8-dimensional Hyperkähler manifold is equipped with a holomorphic volume form Υ which is (4, 0). Also recall that $F_A \in \mathfrak{su}(4) \otimes \Lambda^2 T^* M$, and so if it is wedged with a regular (l, m)-form, only the cotangent component interacts, leaving the Lie-algebra unaffected. Therefore, we can see that in HYM we have

$$F_A \wedge \Upsilon = 0. \tag{58}$$

In particular $F_A \wedge \Re(\Upsilon) = 0$, thus we retrieve the equation 51 of the HYM-instanton. We therefore see that the interplay of $Spin(7)$ and Hyperkähler geometry extends to the instantons.

6.8 Lagrangian instantons

The last remaining case to consider is the Lagrangian calibration $\Re(e^{i\theta}\Upsilon)$. Note that this is gain only possible for dimension 8, since only then Υ is a $n-4$ form. The corresponding generalised Yang-Mills equation would be then:

$$\Re(e^{i\theta}\Upsilon) \wedge F_A = - \star F_A. \tag{59}$$

We have seen that if the bundle admits a Yang-Mills connection, it forces $\Upsilon \wedge F_A = 0$, thus rendering the eq. 59 without solution for any real θ. Thus in this case there are no Lagrangian instantons. The Donaldson-Uhlenbeck-Yau, allows to readily formulate the formal result:

Proposition 4. *Let M and E be a 8-dimensional Calabi-Yau manifold and bundle as in theorem 12. Then the bundle admits no Lagrangian instantons.*

This, however does not exclude the possibility of the eq. 59 to have solutions in different conditions. One observation we can make is that the (1, 1) part of the curvature tensor is irrelevant, and thus it is admissible to consider F_A not possessing it.

7 Summary and possible extensions

The analysis of calibrated geometry and gauge theory on dimension 8 Hyperkähler manifolds exposed intimate relations between the two. While many properties of 4 dimensional manifolds are lost in attempts to extend them, such as loss of spherical symmetry of the Calabi metric, or failure of Hawking-Gibbons and LeBrunn ansatze, there exist some present only in dimension 8, including the interplay of Hermitian and $Spin(7)$ instantons. It is also the case, however, that presence of additional structures invokes no new properties, as it happen for the 2-sphere of Kähler forms on Hyperkähler manifolds, which gives neither new calibrated submanifolds, nor instantons. There, however, remains question of general existence of Lagrangian instantons, exclusive still to dimension 8. Also, the mixed-Kähler, problem could be extended for mixed-Lagrangian and mixed-Cayley calibrations in the same manner. However, since both ultimately depend on the Kähler forms, as it has been shown, it is likely the issues persist. Similar analysis, as suggested, for instance, in [15], may be attempted in higher dimensions, although further loss of well-behaved constructions is likely. On the other hand, there was no mention of, say, exceptional G_2 geometry, which is known to admit both calibrations and instantons [12], and, kin to $Spin(7)$ may introduce other inter-structural properties.

References

[1] Silva ACd. Lectures on symplectic geometry. Springer-Verlag; 2001.

[2] Lotay JD. Lecture notes in Riemannian Geometry. Oxford University; 2020-2021.

[3] Tian G. Gauge Theory and Calibrated Geometry, I. The Annals of Mathematics. 2000;151(1):193.

[4] Dancer A. Hyper-Kähler manifolds. Surveys in Differential Geometry. 2001;6(1):15–38.

[5] Karigiannis S. Introduction to G_2 Geometry. Lectures and Surveys on $G2$-Manifolds and Related Topics Fields Institute Communications. 2020:3–50.

[6] Beauville A. Riemannian Holonomy and Algebraic Geometry; 1999.

[7] Calabi E. Métriques kählériennes et fibrés holomorphes. Annales scientifiques de l'École Normale Supérieure. 1979;4e série, 12(2):269–294. Available from: `http://www.numdam.org/articles/10.24033/asens.1367/`.

[8] Gibbons GW, Hawking SW. Classification of Gravitational Instanton Symmetries. Euclidean Quantum Gravity. 1993:562–581.

[9] Auvray H. From ALE to ALF gravitational instantons. Compositio Mathematica. 2018;154(6):1159–1221.

[10] Lebrun C. Complete Ricci-flat Kähler metrics on $\mathbb{C}^n$ need not be flat. Several Complex Variables and Complex Geometry Proceedings of Symposia in Pure Mathematics. 1991:297–304.

[11] Harvey R, Lawson HB. Calibrated geometries. Acta Mathematica. 1982;148:47–157.

[12] Lotay JD. Calibrated Submanifolds. Lectures and Surveys on G2-Manifolds and Related Topics Fields Institute Communications. 2020:69–101.

[13] Lotay JD. Lecture notes in Differentiable Manifolds. Oxford University; 2020-2021.

[14] Bryant R, Harvey R. Submanifolds in Hyper-Kahler Geometry. Journal of the American Mathematical Society. 1989;2(1):1.

[15] Donaldson SK, Thomas RP. Gauge theory in higher dimensions. In: Conference on Geometric Issues in Foundations of Science in honor of Sir Roger Penrose's 65th Birthday; 1996. p. 31–47.

[16] Donaldson SK, Kronheimer PB. The geometry of four-manifolds. Clarendon Press; 1990.

[17] Uhlenbeck K, Yau ST. On the existence of hermitian-yang-mills connections in stable vector bundles. Communications on Pure and Applied Mathematics. 1986;39(S1).

[18] Lewis C. Spin(7) instantons [Ph.D. thesis]. Oxford University; 1998.